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General Solution of Equations of Motion of Axisymmetric Problem of Micro-Isotropic, Micro-Elastic Solid

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ABSTRACT
In this paper, we obtain the general solution of equations of motion of axisymmetric problem of micro-isotropic, micro-elastic solid in static case. The equations of motion of axisymmetric problem are converted into vector matrix differential equations using the Hankel transform. Applying the technique of solving the eigen value problem, the general solution of the said problem is obtained. The results of the corresponding problem in linear micropolar elasticity are obtained as a particular case of this paper.
KEYWORDS: Micro-isotropic, micro-elastic solid, Eigen value, Hankel transform.

I. INTRODUCTION

The classical theory of elasticity describes well the behavior of construction materials provided the stresses do not exceed the elastic limit and no stress concentration occurs. The discrepancy between the results of the classical theory of elasticity and the experiments [1] appears in all the cases when the microstructure of the body is significant. The materials having microstructure are metals, polymers, rocks and concrete. The influence of microstructure is particularly evident in the case of elastic vibrations of high frequency and/or small wave length. To remove the short comings of the classical theory of elasticity, Eringen [2] introduced the theory of micromorphic materials which includes the micromotion. This theory was simplified by Koh [3] extending the concept of coincidence of principal directions of stresses and strain of classical elasticity and named it as the theory of micro-isotropic, micro-elastic materials. Nowacki [4] has shown that the equations of motion of axisymmetric problem of micropolar solid can be decomposed into two mutually independent sets of three equations. Das *et al.* [5, 6] have obtained general solution of equations of motion in thermoelasticity and magnetothermo-elasticity using eigen value approach to solve vector matrix differential equation.

In the present paper, we apply the technique of solving an eigen value problem to obtain the general solution of the axisymmetric problem of micro-isotropic, micro-elastic solid. The results of the corresponding problem in micropolar theory [7] are obtained as a particular case of it.

II. BASIC EQUATIONS

The equations of motion and the constitute equations of micro-isotropic, micro-elastic solid under the absence of body forces and body couples are given by Parameshwaran and Koh [8]
The displacement equations of motion are

$$(A_1+A_2-A_3)\nabla(\nabla.u)+(A_2+A_3)\nabla^2 u \; +2A_3(\nabla\times\phi) \; =\rho\ddot{u} \tag{1}$$

$$2(B_4+B_5)\nabla(\nabla.\phi)+2B_3\nabla^2\phi-2A_3(\nabla\times u)-4A_3\phi=\rho j\ddot{\phi} \tag{2}$$

$$B_1\phi_{pp,kk}\delta_{ij}+2B_2\phi_{(ij),kk}-A_4\phi_{pp}\delta_{ij}-2A_5\phi_{(ij)}=\frac{1}{2}\rho j\ddot{\phi}_{(ij)} \tag{3}$$

The stress, couple-stress and stress moment are as follows.

$$t_{(km)}=A_1e_{pp}\delta_{km}+2A_2e_{km} \tag{4}$$

$$t_{[km]}=\sigma_{[km]}=2A_3\varepsilon_{pkm}(r_p+\phi_p) \tag{5}$$

$$\sigma_{(km)}=-A_4\phi_{pp}\delta_{km}-2A_5\phi_{(km)} \tag{6}$$

$$t_{k(mn)}=B_1\phi_{pp,k}\delta_{mn}+2B_2\phi_{(mn),k} \tag{7}$$

$$m_{kl}=-2(B_3\phi_{l,k}+B_4\phi_{k,l}+B_5\phi_{p,p}\delta_{kl}) \tag{8}$$

where

$$A_1 = \lambda + \sigma_1, \qquad B_1 = \tau_3,$$
$$A_2 = \mu + \sigma_2, \qquad 2B_2 = \tau_7 + \tau_{10},$$
$$A_3 = \sigma_5, \qquad B_3 = 2\tau_4 + 2\tau_9 + \tau_7 - \tau_{10},$$
$$A_4 = -\sigma_1, \qquad B_4 = -2\tau_4,$$
$$A_5 = -\sigma_2, \qquad B_5 = -2\tau_9. \tag{9}$$

subject to the conditions

$$3A_1 + 2A_2 > 0, \quad A_2 > 0, \quad A_3 > 0,$$
$$3A_4 + 2A_5 > 0, \quad A_5 > 0,$$
$$3B_1 + 2B_2 > 0, \quad B_2 > 0,$$
$$B_3 > 0, \quad -B_3 < B_4 < B_3, \quad B_3 + B_4 + B_5 > 0. \tag{10}$$

where ρ is the average mass density, j is the micro-inertia. The macro displacement in the micro elastic continuum is denoted by u_k and the micro deformation by $\phi_{(mn)}$, for the linear theory we have the macro-strain $e_{km} = e_{(k,m)}$, the macro rotation vector $r_k = \frac{1}{2}\varepsilon_{kmn}u_{n,m}$, the micro-strain $\phi_{(mn)}$ and micro-rotation $\phi_p = \frac{1}{2}\varepsilon_{pkm}\phi_{km}$. The stress measures are the asymmetric stress (macro-stress) t_{mn}, the relative stress (micro-stress) σ_{km} and the stress moment t_{kmn} and also the couple stress tensor $m_{kp} = \varepsilon_{pnm}t_{kmn}$. The symbol ( ) appeared in suffix of a quantity indicate that the quantity is symmetric and [ ] shows the quantity is skew-symmetric. λ,μ, σ_1, σ_2, σ_5, τ_3, τ_4, τ_7, τ_9 and τ_{10} are the ten elastic moduli. Further, ε_{pkm} is the permutation symbol and δ_{km} is the Kronecker delta. The (.) denotes the derivatives with respect to time.

III. FORMULATION AND SOLUTION OF THE PROBLEM

The problem is to find the general solutions of axisymmetric equations of static micro-isotropic, micro-elastic material under the absence of body forces and body couples, we take $\ddot{u} = \ddot{\phi} = 0$ and the cylindrical coordinates r, θ and z are introduced.

The equations of motion (1) and (2) for the static case are as follows.

$$(A_2 + A_3)\left[\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right] + (A_1 + A_2 - A_3)\frac{\partial e}{\partial r} + 2A_3\left[\frac{1}{r}\frac{\partial \phi_z}{\partial \theta} - \frac{\partial \phi_\theta}{\partial z}\right] = 0 \tag{11}$$

$$(A_2 + A_3)\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right] + (A_1 + A_2 - A_3)\frac{1}{r}\frac{\partial e}{\partial \theta} + 2A_3\left[\frac{\partial \phi_r}{\partial z} - \frac{\partial \phi_z}{\partial r}\right] = 0 \tag{12}$$

$$(A_2 + A_3)\left[\nabla^2 u_z\right] + (A_1 + A_2 - A_3)\frac{\partial e}{\partial z} + 2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}(r\phi_\theta) - \frac{\partial \phi_r}{\partial \theta}\right] = 0 \tag{13}$$

$$2B_3\left[\nabla^2 \phi_r - \frac{\phi_r}{r^2} - \frac{2}{r^2}\frac{\partial \phi_\theta}{\partial \theta}\right] - 4A_3\phi_r + (2B_4 + 2B_5)\frac{\partial e'}{\partial r} - 2A_3\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right] = 0 \tag{14}$$

$$2B_3\left[\nabla^2 \phi_\theta - \frac{\phi_\theta}{r^2} + \frac{2}{r^2}\frac{\partial \phi_r}{\partial \theta}\right] - 4A_3\phi_\theta + (2B_4 + 2B_5)\frac{1}{r}\frac{\partial e'}{\partial \theta} - 2A_3\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] = 0 \tag{15}$$

$$2B_3\left[\nabla^2 \phi_z\right] - 4A_3\phi_z + (2B_4 + 2B_5)\frac{\partial e'}{\partial z} - 2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right] = 0 \tag{16}$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$e' = \frac{1}{r}\frac{\partial}{\partial r}\left(r\phi_r\right) + \frac{1}{r}\frac{\partial \phi_\theta}{\partial \theta} + \frac{\partial \phi_z}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In case of the vectors of macro-displacement u and micro-rotation ϕ depend only on the coordinates r and z, the equations (11) to (16) reduce to

$$\left(A_2 + A_3\right)\left[\nabla^2 u_r - \frac{u_r}{r^2}\right] + \left(A_1 + A_2 - A_3\right)\frac{\partial e}{\partial r} - 2A_3\left[\frac{\partial \phi_\theta}{\partial z}\right] = 0 \qquad (17)$$

$$\left(A_2 + A_3\right)\left[\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right] + +2A_3\left[\frac{\partial \phi_r}{\partial z} - \frac{\partial \phi_z}{\partial r}\right] = 0 \qquad (18)$$

$$\left(A_2 + A_3\right)\left[\nabla^2 u_z\right] + \left(A_1 + A_2 - A_3\right)\frac{\partial e}{\partial z} + 2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\phi_\theta\right)\right] = 0 \qquad (19)$$

$$2B_3\left[\nabla^2 \phi_r - \frac{\phi_r}{r^2}\right] - 4A_3\phi_r + \left(2B_4 + 2B_5\right)\frac{\partial e'}{\partial r} + 2A_3\left[\frac{\partial u_\theta}{\partial z}\right] = 0 \qquad (20)$$

$$2B_3\left[\nabla^2 \phi_\theta - \frac{\phi_\theta}{r^2}\right] - 4A_3\phi_\theta - 2A_3\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] = 0 \qquad (21)$$

$$2B_3\left[\nabla^2 \phi_z\right] - 4A_3\phi_z + \left(2B_4 + 2B_5\right)\frac{\partial e'}{\partial z} - 2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}\left(ru_\theta\right)\right] = 0 \qquad (22)$$

where

$$e = \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{\partial u_z}{\partial z}$$

$$e' = \frac{1}{r}\frac{\partial}{\partial r}\left(r\phi_r\right) + \frac{\partial \phi_z}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Equations given by (17) to (22) can be split into two sets of equations. One of these is coupled in u_r, u_z, ϕ_θ and other set is coupled in $\phi_r, \phi_\theta, u_\theta$. These two sets are given by the equations (23) to (25) and (26) to (28).

$$\left(A_2 + A_3\right)\left[\nabla^2 u_r - \frac{u_r}{r^2}\right] + \left(A_1 + A_2 - A_3\right)\frac{\partial e}{\partial r} - 2A_3\left[\frac{\partial \phi_\theta}{\partial z}\right] = 0 \qquad (23)$$

$$\left(A_2 + A_3\right)\left[\nabla^2 u_z\right] + \left(A_1 + A_2 - A_3\right)\frac{\partial e}{\partial z} + 2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\phi_\theta\right)\right] = 0 \qquad (24)$$

$$2B_3\left[\nabla^2 \phi_\theta - \frac{\phi_\theta}{r^2}\right] - 4A_3\phi_\theta - 2A_3\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right] = 0 \qquad (25)$$

and

$$2B_3\left[\nabla^2 \phi_r - \frac{\phi_r}{r^2}\right] - 4A_3\phi_r + \left(2B_4 + 2B_5\right)\frac{\partial e'}{\partial r} + 2A_3\left[\frac{\partial u_\theta}{\partial z}\right] = 0 \qquad (26)$$

$$2B_3\left[\nabla^2\phi_z\right]-4A_3\phi_z+\left(2B_4+2B_5\right)\frac{\partial e'}{\partial z}-2A_3\frac{1}{r}\left[\frac{\partial}{\partial r}\left(ru_\theta\right)\right]=0 \tag{27}$$

$$\left(A_2+A_3\right)\left[\nabla^2 u_\theta-\frac{u_\theta}{r^2}\right]++2A_3\left[\frac{\partial\phi_r}{\partial z}-\frac{\partial\phi_z}{\partial r}\right]=0 \tag{28}$$

The equations (23), (24) and (25) are three mutually independent functions u_r, u_z and ϕ_θ involved. Multiplying (24) by $rJ_0(\xi r)$ and (23), (25) by $rJ_1(\xi r)$ and integrating between the limits 0 to ∞ we find that the system of partial differential equations (23) to (25) reduces to the following system of ordinary differential equations.

$$\left[\left\{-\left(A_2+A_3\right)\right\}D^2-\left\{-\left(A_1+2A_2\right)\right\}\xi^2\right]\overline{u_r}-\left(-A_1-A_2+A_3\right)\xi D\overline{u_z}-2A_3D\overline{\phi_\theta}=0 \tag{29}$$

$$\left(-A_1-A_2+A_3\right)\xi D\overline{u_r}+\left[\left\{-\left(A_1+2A_2\right)\right\}D^2-\left\{-\left(A_2+A_3\right)\right\}\xi^2\right]\overline{u_z}+2A_3\xi\overline{\phi_\theta}=0 \tag{30}$$

$$2A_3D\overline{u_r}+2A_3\xi\overline{u_z}+\left[2B_3\left(D^2-\xi^2\right)-4A_3\right]\overline{\phi_\theta}=0 \tag{31}$$

where $\overline{u_r}, \overline{u_z}$ and $\overline{\phi_\theta}$ are the Hankel transforms of the functions $-u_r, -u_z$ and ϕ_θ respectively and are given by

$$\overline{u_r}=-\int_0^\infty ru_rJ_1(\xi r)dr,\ \overline{u_z}=-\int_0^\infty ru_zJ_0(\xi r)dr,\ \overline{\phi_\theta}=\int_0^\infty r\phi_\theta J_1(\xi r)dr.$$

Further, $D=\frac{d}{dz}$ and $D^2=\frac{d^2}{dz^2}$.

We represent the equations (29) to (31) as a matrix differential equation

$$\left[PD^2-QD-R\right]X=0 \tag{32}$$

where

$$P=\begin{bmatrix}-\left(A_2+A_3\right) & 0 & 0\\ 0 & -\left(A_1+2A_2\right) & 0\\ 0 & 0 & 2B_3\end{bmatrix},\ Q=\begin{bmatrix}0 & \left(-A_1-A_2+A_3\right)\xi & 2A_3\\ -\left(-A_1-A_2+A_3\right)\xi & 0 & 0\\ -2A_3 & 0 & 0\end{bmatrix}$$

$$R=\begin{bmatrix}-\left(A_1+2A_2\right)\xi^2 & 0 & 0\\ 0 & -\left(A_2+A_3\right)\xi^2 & -2A_3\xi\\ 0 & -2A_3\xi & 2B_3\xi^2+4A_3\end{bmatrix} \text{ and } X=\begin{bmatrix}\overline{u_r}\\ \overline{u_z}\\ \overline{\phi_\theta}\end{bmatrix}.$$

We suppose

$$X'=DX=D\begin{bmatrix}\overline{u_r}\\ \overline{u_z}\\ \overline{\phi_\theta}\end{bmatrix},\ X''=D^2X=D^2\begin{bmatrix}\overline{u_r}\\ \overline{u_z}\\ \overline{\phi_\theta}\end{bmatrix} \tag{33}$$

Now the equation (32) can be expressed as

$$PX''-QX'-RX=0 \tag{34}$$

Multiplying the equation (34) by P^{-1} we have

$$X''=L_1X'+L_2X \tag{35}$$

where

$$L_1 = P^{-1}Q = \begin{bmatrix} 0 & \dfrac{-(-A_1 - A_2 + A_3)\xi}{(A_2 + A_3)} & \dfrac{-2A_3}{(A_2 + A_3)} \\ \dfrac{(-A_1 - A_2 + A_3)\xi}{(A_1 + 2A_2)} & 0 & 0 \\ \dfrac{-A_3}{B_3} & 0 & 0 \end{bmatrix}$$

$$L_2 = P^{-1}R = \begin{bmatrix} \dfrac{(A_1 + 2A_2)\xi^2}{(A_2 + A_3)} & 0 & 0 \\ 0 & \dfrac{(A_2 + A_3)\xi^2}{(A_1 + 2A_2)} & \dfrac{2A_3\xi}{(A_1 + 2A_2)} \\ 0 & \dfrac{-A_3\xi}{B_3} & \dfrac{(B_3\xi^2 + 2A_3)}{B_3} \end{bmatrix}$$

We express equations (33) and (35) as a single matrix differential equation

$$\frac{d}{dz}\begin{bmatrix} X' \\ X \end{bmatrix} = \begin{bmatrix} L_1 & L_2 \\ I & O \end{bmatrix}\begin{bmatrix} X' \\ X \end{bmatrix} \tag{36}$$

where I and O are unit and zero matrices of order 3 respectively.

Suppose $E = \begin{bmatrix} L_1 & L_2 \\ I & O \end{bmatrix}$ and $Y = \begin{bmatrix} X' \\ X \end{bmatrix}$ (37)

In view of equation (37) the equation (36) can be written as

$$\frac{d}{dz}Y = EY \tag{38}$$

where

$$E = \lfloor e_{ij} \rfloor_{6x6}$$

Assume $Y = We^{sz}$ be a solution of equation (38) then we have $EW = sW$ (39)

$$e_{12} = \frac{-(-A_1 - A_2 + A_3)\xi}{(A_2 + A_3)},\ e_{13} = \frac{-2A_3}{(A_2 + A_3)},\ e_{14} = \frac{(A_1 + 2A_2)\xi^2}{(A_2 + A_3)}$$

$$e_{21} = \frac{(-A_1 - A_2 + A_3)\xi}{(A_1 + 2A_2)},\ e_{25} = \frac{(A_2 + A_3)\xi^2}{(A_1 + 2A_2)},\ e_{26} = \frac{2A_3\xi}{(A_1 + 2A_2)}$$

$$e_{31} = \frac{-A_3}{B_3},\ e_{35} = \frac{-A_3\xi}{B_3},\ e_{36} = \frac{B_3\xi^2 + 2A_3}{B_3}.$$

Hence, s is an eigen value of the matrix E and W is the corresponding eigen vector.

The eigen values of the matrix E are the roots of the equation

$$\det(E - sI) = 0 \tag{40}$$

$$i.e.,\begin{vmatrix} -s & e_{12} & e_{13} & e_{14} & 0 & 0 \\ e_{21} & -s & 0 & 0 & e_{25} & e_{26} \\ e_{31} & 0 & -s & 0 & e_{35} & e_{36} \\ 1 & 0 & 0 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & -s & 0 \\ 0 & 0 & 1 & 0 & 0 & -s \end{vmatrix} = 0 \tag{41}$$

Expanding the determinant of equation (41) we get the characteristic equation as

$$s^6 - \left(2\xi^2 + \zeta^2\right)s^4 + \left(\xi^4 + 2\xi^2\zeta^2\right)s^2 - \xi^4\zeta^2 = 0 \tag{42}$$

where

$$\zeta^2 = \xi^2 + l^2 \text{ and } l^2 = \frac{2A_3\left(A_2 + 2A_3\right)}{B_3\left(A_2 + A_3\right)}$$

The eigen values of the matrix E are the roots of the equation (42) and they are $\pm\xi, \pm\xi, \pm\zeta$.

Let $s_1 = \xi, s_2 = -\xi, s_3 = \zeta$ and $s_4 = -\zeta$

The four eigen vectors corresponding to four distinct eigen values s_1, s_2, s_3, s_4 of the matrix E are obtained by solving the following homogeneous equations.

$$\begin{bmatrix} -s & e_{12} & e_{13} & e_{14} & 0 & 0 \\ e_{21} & -s & 0 & 0 & e_{25} & e_{26} \\ e_{31} & 0 & -s & 0 & e_{35} & e_{36} \\ 1 & 0 & 0 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & -s & 0 \\ 0 & 0 & 1 & 0 & 0 & -s \end{bmatrix} \begin{bmatrix} W_1(s) \\ W_2(s) \\ W_3(s) \\ W_4(s) \\ W_5(s) \\ W_6(s) \end{bmatrix} = 0 \tag{43}$$

for $s = s_1, s_2, s_3, s_{4.}$

If we denote the cofactors of the elements of the first row of the coefficient matrix of the equation (43) by $E_i(s)$, for $i = 1,2,3,4,5,6$ then

$$W(s) = \left[E_1(s) \quad E_2(s) \quad E_3(s) \quad E_4(s) \quad E_5(s) \quad E_6(s)\right]^T \tag{44}$$

are the solutions of the equation (43) and hence they are eigen vectors corresponding to the eigen values s_1, s_2, s_3, s_4 of the matrix E. The elements of equation (44) are given by

$$E_1(s) = -s^5 + s^3\left[\frac{\left(A_1 + 3A_2 + A_3\right)}{\left(A_1 + 2A_2\right)}\xi^2 + \frac{2A_3}{B_3}\right] + s\left[\frac{\left(A_2 + A_3\right)}{\left(A_1 + 2A_2\right)}\xi^4 + \frac{2A_3\left(A_2 + 2A_3\right)}{B_3\left(A_1 + 2A_2\right)}\xi^2\right]$$

$$E_2(s) = s^2\xi\left[\frac{\left(A_1 + A_2 - A_3\right)}{\left(A_1 + 2A_2\right)}\left(s^2 - \xi^2\right) - \frac{2A_3\left(A_1 + A_2 - 2A_3\right)}{B_3\left(A_1 + 2A_2\right)}\right]$$

$$E_3(s) = A_3 s^2\left[\frac{s^2 - \xi^2}{B_3}\right]$$

$$E_4(s) = -s^4 + s^2\left[\frac{\left(A_1 + 3A_2 + A_3\right)}{\left(A_1 + 2A_2\right)}\xi^2 + \frac{2A_3}{B_3}\right] - s^2\xi^2\frac{\left(A_2 + A_3\right)}{\left(A_1 + 2A_2\right)}$$

$$E_5(s) = \left[\frac{\left(A_1 + A_2 - A_3\right)}{\left(A_1 + 2A_2\right)}\left(s^2 - \xi^2\right) - \frac{2A_3\left(A_1 + A_2 - 3A_3\right)}{B_3\left(A_1 + 2A_2\right)}\right]\xi s$$

$$E_6(s) = A_3 s\left[\frac{s^2-\xi^2}{B_3}\right]$$

Thus, the solution of the differential equation (38) is given by Das *et al.* [5]

$$Y = c_1 W(s_1)\exp(s_1 z) + c_2 \frac{d}{dz}\left[W(s)\exp(sz)\right]_{s=s_1} + c_3 W(s_2 z)\exp(s_2 z) + c_4 \frac{d}{dz}\left[W(s)\exp(sz)\right]_{s=s_2}$$
$$+ c_5 W(s_3)\exp(s_3 z) + c_6 W(s_4)\exp(s_4 z) \quad (45)$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary constants which are to be determined from the boundary conditions.

The equation (45) can be expressed as

$$Y = (c_1 + c_2 z)W(s_1)\exp(s_1 z) + c_2 W'(s_1)\exp(s_1 z) + (c_3 + c_4 z)W(s_2)\exp(s_2 z) + c_4 W'(s_2)\exp(s_2 z)$$
$$+ c_5 W(s_3)\exp(s_3 z) + c_6 W(s_4)\exp(s_4 z) \quad (46)$$

where (‘) represents the differentiation with respect to z.
For the half space $z \geq 0$, the equation (46) reduces to the form

$$Y = (c_3 + c_4 z)W(s_2)\exp(s_2 z) + c_4 W'(s_2)\exp(s_2 z) + c_6 W(s_4)\exp(s_4 z) \quad (47)$$

where the constants c_3, c_4, c_6 are to be determined from the boundary conditions.
Equating the corresponding elements of matrices of equation (47), we obtain

$$\overline{u_r'}(z) = [(c_3 + c_4 z)E_1(s_2) + c_4 E_1(s_2)]\exp(s_2 z) + c_6 E_1(s_4)\exp(s_4 z)$$
$$\overline{u_z'}(z) = [(c_3 + c_4 z)E_2(s_2) + c_4 E_2(s_2)]\exp(s_2 z) + c_6 E_{21}(s_4)\exp(s_4 z)$$
$$\overline{\phi_\theta'}(z) = [(c_3 + c_4 z)E_3(s_2) + c_4 E_3(s_2)]\exp(s_2 z) + c_6 E_3(s_4)\exp(s_4 z)$$
$$\overline{u_r}(z) = [(c_3 + c_4 z)E_4(s_2) + c_4 E_4(s_2)]\exp(s_2 z) + c_6 E_4(s_4)\exp(s_4 z) \quad (48)$$
$$\overline{u_z}(z) = [(c_3 + c_4 z)E_5(s_2) + c_4 E_5(s_2)]\exp(s_2 z) + c_6 E_5(s_4)\exp(s_4 z) \quad (49)$$
$$\overline{\phi_\theta}(z) = [(c_3 + c_4 z)E_6(s_2) + c_4 E_6(s_2)]\exp(s_2 z) + c_6 E_6(s_4)\exp(s_4 z) \quad (50)$$

The equations (26), (27) and (28) three mutually independent functions u_r, u_z and ϕ_θ are involved. Multiplying (27) by $rJ_0(\xi r)$ and (26), (28) by $rJ_1(\xi r)$ and integrating between the limits 0 to ∞. We find that the system of partial differential equations (26) to (28) reduces to the following system of ordinary differential equations.

$$\left[2B_3 D^2 - 2(B_3 + B_4 + B_5)\xi^2 - 4A_3\right]\overline{\phi_r} - 2(B_4 + B_5)\xi D\overline{\phi_z} - 2A_3 D\overline{u_\theta} = 0 \quad (51)$$
$$2(B_4 + B_5)\xi D\overline{\phi_r} + \left[2(B_3 + B_4 + B_5)D^2 - 2B_3\xi^2 - 4A_3\right]\overline{\phi_z} + 2A_3\xi\overline{u_\theta} = 0 \quad (52)$$
$$2A_3 D\overline{\phi_r} + 2A_3\xi\overline{\phi_z} + (-A_2 - A_3)(D^2 - \xi^2)\overline{u_\theta} = 0 \quad (53)$$

where $\overline{\phi_r}, \overline{\phi_z}$ and $\overline{u_\theta}$ are the Hankel transforms of the functions $-\phi_r, -\phi_z$ and u_θ respectively and are given by

$$\overline{\phi_r} = -\int_0^\infty r\phi_r J_1(\xi r)dr,\ \overline{\phi_z} = -\int_0^\infty r\phi_z J_0(\xi r)dr,\ \overline{u_\theta} = \int_0^\infty r u_\theta J_1(\xi r)dr.$$

We repeat the method adopted for solving the first set of equations. The equations (51) to (53) can be expressed as vector matrix differential equation.

$$\frac{d}{dz}Y = FY \quad (54)$$

where

$$Y = \begin{bmatrix} \overline{\phi_r}' & \overline{\phi_z}' & \overline{u_\theta}' & \overline{\phi_r} & \overline{\phi_z} & \overline{u_\theta} \end{bmatrix}^T \text{ and } F = \lfloor f_{ij} \rfloor \quad (55)$$

The elements of the matrix F are given by

$$f_{11} = f_{15} = f_{16} = 0,\ f_{12} = \frac{(B_4 + B_5)\xi}{B_3},\ f_{13} = \frac{A_3}{B_3},\ f_{14} = \frac{(B_3 + B_4 + B_5)\xi^2 + 2A_3}{B_3}.$$

$$f_{22} = f_{23} = f_{24} = 0,\ f_{21} = \frac{-(B_4 + B_5)\xi}{(B_3 + B_4 + B_5)},\ f_{25} = \frac{A_3\xi^2 + 2A_3}{(B_3 + B_4 + B_5)},\ f_{26} = \frac{-A_3\xi}{(B_3 + B_4 + B_5)}.$$

$$f_{32} = f_{33} = f_{34} = 0,\ f_{31} = \frac{2A_3}{(A_2 + A_3)},\ f_{35} = \frac{2A_3\xi}{(A_2 + A_3)},\ f_{36} = \xi^2.$$

$$f_{41} = 1,\ f_{42} = f_{43} = f_{44} = f_{45} = f_{46} = 0.$$

$$f_{52} = 1,\ f_{51} = f_{53} = f_{54} = f_{55} = f_{56} = 0.$$

$$f_{63} = 1,\ f_{61} = f_{62} = f_{64} = f_{65} = f_{66} = 0. \tag{56}$$

Again, if g is an eigen value of the matrix F then $Y = U\exp(gz)$ is a solution of equation (54). Hence, U is the corresponding eigen vector.

The eigen values of F are the roots of the characteristic equation

$$|F - gI| = 0 \tag{57}$$

$$\begin{vmatrix} -g & f_{12} & f_{13} & f_{14} & 0 & 0 \\ f_{21} & -g & 0 & 0 & f_{25} & f_{26} \\ f_{31} & 0 & -g & 0 & f_{35} & f_{36} \\ 1 & 0 & 0 & -g & 0 & 0 \\ 0 & 1 & 0 & 0 & -g & 0 \\ 0 & 0 & 1 & 0 & 0 & -g \end{vmatrix} = 0 \tag{58}$$

Now simplifying equation (58) and using equation (56) therein we obtain the characteristic equation as

$$g^6 - g^4\left(\xi^2 + \lambda_1^2 + \lambda_2^2\right) + g^2\left(\xi^2\lambda_1^2 + \xi^2\lambda_2^2 + \lambda_1^2\lambda_2^2\right) - \xi^2\lambda_1^2\lambda_2^2 = 0 \tag{59}$$

where

$$\lambda_1^2 = \xi^2 + k_1^2,\ \lambda_2^2 = \xi^2 + k_2^2,\ k_1^2 = \frac{A_3}{(B_3 + B_4 + B_5)},\ k_2^2 = \frac{2A_3(A_2 + 2A_3)}{(A_2 + A_3)B_3}.$$

The roots are $\pm\xi, \pm\lambda_1, \pm\lambda_2$ which are the distinct eigen values of the matrix F.

The corresponding eigen vectors are obtained by solving the following homogeneous equation.

$$\begin{bmatrix} -g & f_{12} & f_{13} & f_{14} & 0 & 0 \\ f_{21} & -g & 0 & 0 & f_{25} & f_{26} \\ f_{31} & 0 & -g & 0 & f_{35} & f_{36} \\ 1 & 0 & 0 & -g & 0 & 0 \\ 0 & 1 & 0 & 0 & -g & 0 \\ 0 & 0 & 1 & 0 & 0 & -g \end{bmatrix} \begin{bmatrix} U_1(g) \\ U_2(g) \\ U_3(g) \\ U_4(g) \\ U_5(g) \\ U_6(g) \end{bmatrix} = 0 \tag{60}$$

for $g = g_i, i = 1,2,3,4,5,6.$

here $g_1 = \xi, g_2 = -\xi, g_3 = \lambda_1, g_4 = -\lambda_1, g_5 = \lambda_2, g_6 = -\lambda_2.$

We denote the cofactors of the elements of the first row of the coefficient matrix in equation (60) by $F_i(g), i = 1,2,3,4,5,6.$ then

$U(g) = [F_1(g) \quad F_2(g) \quad F_3(g) \quad F_4(g) \quad F_5(g) \quad F_6(g)]^T$ are the solutions of the equation (60) and hence they are eigen vectors corresponding to the eigen values $g_i, i = 1,2,3,4,5,6$ of the matrix F.

The expressions of $F_i(g), i=1,2,3,4,5,6$ are

$$F_1(g)=-g^5+g^3\left[\frac{\xi^2(B_4+B_5+2B_3)+2A_3}{(B_4+B_5+B_3)}\right]-g\xi^2\left[\frac{(A_2+A_3)B_3\xi^2+2A_3(A_2+2A_3)}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

$$F_2(g)=g^4\xi\frac{(B_4+B_5)}{(B_4+B_5+B_3)}-g^2\xi\left[\frac{(B_4+B_5)(A_2+A_3)\xi^2-2A_3^2}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

$$F_3(g)=-g^4\frac{2A_3}{(A_2+A_3)}+g^2\left[\frac{2A_3\xi^2(-B_5+B_3-B_4)+2A_3^2}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

$$F_4(g)=-g^4+g^2\left[\frac{(B_5+2B_3+B_4)\xi^2+2A_3}{(B_4+B_5+B_3)}\right]-\xi^2\left[\frac{B_3(A_2+A_3)\xi^2+2A_3(A_2+2A_3)}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

$$F_5(g)=g^3\xi\frac{(B_4+B_5)}{(B_4+B_5+B_3)}-g\left[\frac{(B_4+B_5)\xi^3}{(B_4+B_5+B_3)}-\frac{2A_3^2\xi}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

$$F_6(g)=-g^3\frac{(2A_3)}{(A_2+A_3)}+g\left[\frac{2A_3\xi^2(-B_5+B_3-B_4)2A_3^2}{(B_4+B_5+B_3)(A_2+A_3)}\right]$$

Since g_1, g_2, g_3, g_4, g_5, g_6 are the distinct roots of characteristic equation (59).

The general solution of the differential equation (55) is given by

$$Y=C_1U(g_1)\exp(g_1z)+C_2U(g_2)\exp(g_2z)+C_3U(g_3)\exp(g_3z)+C_4U(g_4)\exp(g_4z)+ C_5U(g_5)\exp(g_5z)+C_6U(g_6)\exp(g_6z) \tag{61}$$

where C_i , i=1,2,3,4,5,6 are arbitrary constants.

From equation (61), we have

$$\overline{\phi_r'}(z)=\sum_{i=1}^{6}C_iF_1(g_i)\exp(g_iz)$$

$$\overline{\phi_z'}(z)=\sum_{i=1}^{6}C_iF_2(g_i)\exp(g_iz)$$

$$\overline{u_\theta'}(z)=\sum_{i=1}^{6}C_iF_3(g_i)\exp(g_iz)$$

$$\overline{\phi_r}(z)=\sum_{i=1}^{6}C_iF_4(g_i)\exp(g_iz) \tag{62}$$

$$\overline{\phi_z}(z)=\sum_{i=1}^{6}C_iF_5(g_i)\exp(g_iz) \tag{63}$$

$$\bar{u}_\theta(z)=\sum_{i=1}^{6}C_iF_6(g_i)\exp(g_iz) \tag{64}$$

Now, we consider the equations of motion corresponding to micro-strains.
The equations of motion under the absence of body forces and couples the equation (3) involving micro-strains can be expressed as

$$B_1\nabla^2\phi_{pp}+2B_2\nabla^2\phi_{rr}-A_4\phi_{pp}-2A_5\phi_{rr}=0 \tag{65}$$

$$B_1\nabla^2\phi_{pp}+2B_2\nabla^2\phi_{\theta\theta}-A_4\phi_{pp}-2A_5\phi_{\theta\theta}=0 \tag{66}$$

$$B_1\nabla^2\phi_{pp}+2B_2\nabla^2\phi_{zz}-A_4\phi_{pp}-2A_5\phi_{zz}=0 \tag{67}$$

$$2B_2\nabla^2\phi_{(r\theta)} - 2A_5\phi_{(r\theta)} = 0 \tag{68}$$

$$2B_2\nabla^2\phi_{(rz)} - 2A_5\phi_{(rz)} = 0 \tag{69}$$

$$2B_2\nabla^2\phi_{(\theta z)} - 2A_5\phi_{(\theta z)} = 0 \tag{70}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \phi_{pp} = \phi_{rr} + \phi_{\theta\theta} + \phi_{zz}$$

Adding equations (65), (66) and (67) we have

$$(3B_1 + 2B_2)\nabla^2\phi_{pp} - (3A_4 + 2A_5)\phi_{pp} = 0 \tag{71}$$

Subtract equation (66) from equation (65) and subtract equation (67) from equation (66) we get

$$2B_2\nabla^2(\phi_{rr} - \phi_{\theta\theta}) - 2A_5(\phi_{rr} - \phi_{\theta\theta}) = 0 \tag{72}$$

$$2B_2\nabla^2(\phi_{\theta\theta} - \phi_{zz}) - 2A_5(\phi_{\theta\theta} - \phi_{zz}) = 0 \tag{73}$$

From equation (68) we have

$$\nabla^2\phi_{(r\theta)} - m_1^2\phi_{(r\theta)} = 0 \tag{74}$$

where

$$m_1^2 = \frac{A_5}{B_2}$$

The general solution of equation (74) is given by

$$\phi_{(r\theta)}(r,z) = [a_1 I_0(m_1 r) + a_2 K_0(m_1 r)][b_1\cos(nz) + b_2\sin(nz)] \tag{75}$$

As the equations (69) and (70) are similar to equation (68). Hence the solutions of $\phi_{(rz)}$ and $\phi_{(\theta z)}$ are given by

$$\phi_{(rz)}(r,z) = [a_3 I_0(m_1 r) + a_4 K_0(m_1 r)][b_3\cos(nz) + b_4\sin(nz)] \tag{76}$$

$$\phi_{(\theta z)}(r,z) = [a_5 I_0(m_1 r) + a_6 K_0(m_1 r)][b_5\cos(nz) + b_6\sin(nz)] \tag{77}$$

where $I_0(m_1 r)$ and $K_0(m_1 r)$ are modified Bessel function of the first and second kinds respectively.

From equation (72) we have

$$\nabla^2(\phi_{rr} - \phi_{\theta\theta}) - m_1^2(\phi_{rr} - \phi_{\theta\theta}) = 0 \tag{78}$$

The general solution of equation (78) is given by

$$(\phi_{rr} - \phi_{\theta\theta})(r,z) = [a_7 I_0(m_1 r) + a_8 K_0(m_1 r)][b_7\cos(nz) + b_8\sin(nz)] \tag{79}$$

Similarly, the general solution of equation (73) is given by

$$(\phi_{\theta\theta} - \phi_{zz})(r,z) = [a_9 I_0(m_1 r) + a_{10} K_0(m_1 r)][b_9\cos(nz) + b_{10}\sin(nz)] \tag{80}$$

From equation (71) we have

$$\nabla^2\phi_{pp} - m_2^2\phi_{pp} = 0 \tag{81}$$

where

$$m_2^2 = \frac{(3A_4 + 2A_5)}{(3B_1 + 2B_2)}$$

The general solution of equation (81) is given by

$$\phi_{pp}(r,z) = [a_{11} I_0(m_2 r) + a_{12} K_0(m_2 r)][b_{11}\cos(nz) + b_{12}\sin(nz)] \tag{82}$$

Now, solving equations (79), (80) and (82) we get

$$\phi_{rr}(r,z) = [a_7 I_0(m_1 r) + a_8 K_0(m_1 r)][b_7\cos(nz) + b_8\sin(nz)] + [a_9 I_0(m_1 r) + a_{10} K_0(m_1 r)]$$
$$[b_9\cos(nz) + b_{10}\sin(nz)] + [a_{11} I_0(m_2 r) + a_{12} K_0(m_2 r)][b_{11}\cos(nz) + b_{12}\sin(nz)] \tag{83}$$

$$\phi_{\theta\theta}(r,z) = [a_7 I_0(m_1 r) + a_8 K_0(m_1 r)][b_7\cos(nz) + b_8\sin(nz)] \tag{84}$$

$$\phi_{zz}(r,z) = [a_7 I_0(m_1 r) + a_8 K_0(m_1 r)][b_7 \cos(nz) + b_8 \sin(nz)] + [a_9 I_0(m_1 r) + a_{10} K_0(m_1 r)]$$
$$[b_9 \cos(nz) + b_{10} \sin(nz)] \quad (85)$$

here a_0 to a_{12} and b_0 to b_{12} are arbitrary constants.

The arbitrary constants involved in equations (48), (49), (50), (62), (63), (64), (83), (84), (85), (75), (76) and (77) can be determined using specified boundary conditions of a particular problem. Once, these constants are found, it is possible to find displacements, micro-rotation, stress and mirco-streses. The results of micropolar theory Mahalanabis and Manna [7] are obtained as particular case of this paper when $2B_4 = \alpha, 2B_5 = \beta, 2B_3 = \gamma,\ 2A_3 = \kappa\ , A_1 + 2A_3 = \lambda,\ \ A_2 - A_3 = \mu$ and $B_1, B_2, A_4, A_5 \to 0.$

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